Model Based Pose Estimator Using Linear-Programming [*]

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Abstract

*Given a $3D$ object and some measurements for points in this object, it is desired to find the $3D$ location of the object. A new model based pose estimator from stereo pairs based on linear programming (*LP*) is presented. In the presence of outliers, the new* LP *estimator provides better results than maximum likelihood estimators such as weighted least squares, and is usually almost as good as robust estimators such as* LMEDS*. In the presence of noise the new* LP *estimator provides better results than robust estimators such as* LMEDS*, and is slightly inferior to maximum likelihood estimators such as weighted least squares. In the presence of noise and outliers - especially for wide angle stereo - the new estimator provides the best results.*

The LP *estimator is based on correspondence of a points to convex polyhedrons. Each points corresponds to a unique polyhedron, which represents its uncertainty in $3D$ as computed from the stereo pair. Polyhedron can also be computed for $2D$ data point by using a-priori depth boundaries.*

The LP *estimator is a single phase (no separate outlier rejection phase) estimator solved by single iteration (no re-weighting), and always converges to the global minimum of its error function. The estimator can be extended to include random sampling and re-weighting within the standard frame work of a linear program.*

1 Introduction

Model based pose estimation is a well studied problem in computer vision and in photogrametry, where it is called: *absolute orientation*. The objective of the problem is finding the exact location of a known object in $3D$ space from image measurements.

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Numerous references regarding pose estimation appear in the literature, see [16, 8, 19].
Pose estimation problem consists of several sub-problems, including:

1. Feature type selection - points and lines are commonly used [3, 18].

2. Measurement type selection - $3D$ features to $3D$ features, $2D$ features to $3D$ features or a combination of $2D$ features and $3D$ features [7, 9, 10].

3. Estimator selection - least squares, kalman filter, hough transform [9, 10, 12, 1].

This paper is focused on the estimator part for measurements obtained by a stereo head. The proposed estimator is based on point to polyhedron correspondence using linear programming. The estimator was tested for non-optimal conditions including strong non-Gaussian noise, outliers, and wide field of view (up to $\pm 80^o$). The proposed linear-programming (LP) estimator is compared to the following estimators:

1. Least squares - L_2 based estimator (LSQ).

2. Weighted least squares (WLS) using covariance matrices.

3. Least absolute differences - L_1 based estimator (LAD)

4. Least median of squares (LMEDS or LMS)

5. TUKEY BI-WEIGHT M-ESTIMATOR which uses the LMEDS solution as its initial guess.

6. A variant of the TUKEY M-ESTIMATOR that uses covariance matrices as well.

Next we briefly describe each of the above estimators. Section 2 describes the proposed LP estimator. Section 4 describes the testing procedure and test results. Section 5 describes future enhancements of the proposed estimator. The following notation is used: given two sets of points in $3D$, $\{M_i\}$ - the model points and $\{P_i\}$ - the measurements we want to find a rigid $3D$ transformation: a rotation matrix R and a translation vector T that minimizes the distance between $(T + RP)$ and M.

1.1 Least square pose-estimator

The error function is $\sum_i ||T + RP_i - M_i||_2$. To find the transformation we define $S_i = P_i - \bar{P}$, $Q_i = M_i - \bar{M}$ were $\bar{P}$, $\bar{M}$ are the averages of $\{P_i\}$ and $\{M_i\}$ respectively. If $U\Sigma V^\top = SVD(SQ^\top)$ then the rotation matrix R is given by $R = UV^\top$ and the translation vector T is given by $T = \bar{M} - R\bar{P}$. See [5]. Since we would like to use a common framework for our comparison for non least-squares estimators as well, we use the following three stage algorithm and present its result for LSQ as well.

1. We seek an affine transformation F which minimizes some error function. For LSQ the error function is: $\sum_i ||(M_i - FP_i)||_2$. For LSQ F is recovered by $F = (A^\top A)^\# A^\top b$ where: $A = [M_1^\top, \ldots M_n^\top]^\top$, $b = P^\top$ and '#' denotes pseudo inverse.

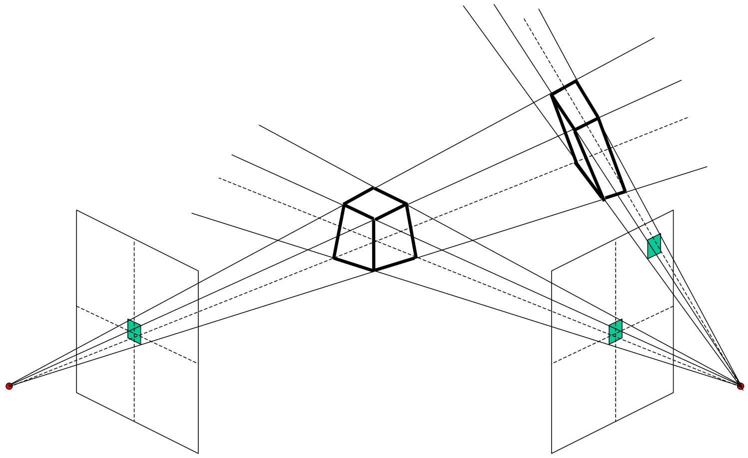

Figure 1. Stereo uncertainty polyhedron - The shape of each polyhedron is a function of the point-rig geometry and the radius of error in the image plane.

2. The rotation part of F denoted as F_r is projected onto the closest (in the least-squares sense) rotation matrix $R = UV^\top$ where $U\Sigma V^\top = SVD(F_r)$. The determinant of R is checked to eliminate reflections ($det(R) = -1$). The step is common for all estimators.

3. The translation is then recovered. For LSQ by $T = \bar{M} - R\bar{P}$. For other estimators by warping the model according to R and then solving for translation.

1.2 Weighted least squares estimator

In WLS we seek an affine transformation F that minimizes: $\sum_i (b - AF)^\top W(b - AF)$. F is recovered by $F = (A^\top WA)^{\#} A^\top Wb$. Where W, the weight matrix is a block diagonal matrix, each block is a 3×3 covariance matrix W_i of the corresponding pair (M_i, P_i). W_i is computed as follows: for p_i, q_i - the projection of the (unknown) $3D$ point P_i onto the two stereo images, we use the bounding rectangles $[(p_i)_x \pm r, (p_i)_y \pm r]$, $[(q_i)_x \pm r, (q_i)_y \pm r]$, to compute a 3D bounding polyhedron D for P_i. The weight matrix W_i is taken as the covariance matrix of the eight vertices of D_i with respect to X, Y, and Z coordinates. See Fig. 1. The radius r is a tunable parameter which will be discussed later in the article. The shape of the polyhedron is dependent upon point location with respect to the stereo head and the image uncertainty radius. This shape varies from point to point and it is not symmetric even though the image uncertainty radius r is symmetric. For this reason a simple least squares does not provide the best estimation.

1.3 Least of absolute differences estimator

In LAD we seek an affine transformation F that minimizes the error function: $\sum_i \|(M_i - FP_i)\|_1$. F is recovered by solving the following linear programming problem:

$$Min : \sum_i (r_i^+)_x + (r_i^-)_x + (r_i^+)_y + (r_i^-)_y + (r_i^+)_z + (r_i^-)_z \quad (1)$$

$$\begin{gathered} Subject\ to \\ (M_i - FP_i)_x \ + \ (r_i^+)_x - (r_i^-)_x = 0 \\ (M_i - FP_i)_y \ + \ (r_i^+)_y - (r_i^-)_y = 0 \\ (M_i - FP_i)_z \ + \ (r_i^+)_y - (r_i^-)_z = 0 \\ r_i^+, r_i^- \geq 0 \end{gathered}$$

In linear programming all variables are non-negative, therefore a real typed variable x is represented by a pair of non negative variables: $x = (x^+ - x^-)$. This holds for the elements of F as well (not explicitly shown) and for the *residual error slack variables* $(r_i)_{x,y,z}$.

At the *global minimum point* we get either $r^+ = 0$ *or* $r^- = 0$ for each pair, and hence $\sum (r^+ + r^-)$ is the sum of least absolute differences [2]. LAD has been successfully used for motion recovery in the presence of noise and outliers, however since the LAD error function is symmetric, and the uncertainty shape in 3D is not, the LAD is not fully suitable to the pose estimation problem. LAD, like other M-estimators, has a breakdown point of zero due to *leverage points*, therefore it is not considered a robust estimator. In the experiments we used strong outliers - up to the maximum possible range within image boundaries, trying to break the LAD estimator by leverage points. Although we managed to break it down when the number of outliers exceeded a certain point (above 40%) we did not see leverage point symptoms. This result complies with earlier results [*]. We believe that this is due to the fact that the error was bounded by the image size.

1.4 Least median of squares estimator

In LMEDS we seek an affine transformation F that minimizes: $median(\|M_i - FP_i\|_2)$. LMEDS is a robust estimator in the sense of its breakdown point which is 0.5 - the largest value possible. Unfortunately, deterministic algorithms for LMEDS have exponential time complexity. To solve this problem a probabilistic algorithm by random sampling is used: several model estimations are recovered by different random samples. The residual error is computed for each estimation and the estimation with the lowest median of squares is selected as the LMEDS estimation. This probability of the algorithm's success is:

$$P = 1 - \left[1 - (1 - q)^k\right]^n \quad (2)$$

where: q is the probability of choosing an outlier. k is the number of guesses needed for model recovery and n is the number of iterations. However, this analysis assumes that the data points are dichotomic - each data point is either an outlier, or a perfectly good inlier. This assumption does not hold in the presence of noise and it affects the accuracy of the algorithm. Note that the time complexity of the probabilistic algorithm grows exponentially with k, thus trying to reduce the noise influence by enlarging the support is very costly. See [14, 15, 4, 11, 17].

1.5 Tukey Bi-weight M-estimator

In TUKEY M-ESTIMATOR we seek an affine transformation F that minimizes:

$$\sum_i \rho\left(\|M_i - FP_i\|/\sigma_i\right) \tag{3}$$

$$\rho(u) = \begin{cases} \frac{b^2}{6}[1-(1-(\frac{u}{b})^2)^3] & |u| \leq b \\ \frac{b^2}{6} & |u| > b \end{cases}$$

Where $\rho(u)$ is the loss function, b is a tuning parameter and σ_i the scale associate with the value of the residual error $r_{i,F} = \|M_i - FP_i\|$ of the inliers. Equation 3 is often solved by an "iterative re-weighted least-squares" [13] with the following weight function:

$$w(u = r_{i,F}/\hat{\sigma}) = \psi(u)/u \tag{4}$$

$$\psi(u) = \rho(u)' = \begin{cases} u[1-(\frac{u}{b})^2]^2 & |u| \leq b \\ 0 & |u| > b \end{cases}$$

Where $\hat{\sigma}$ is a scale estimate. The following scale estimation was used in the test

$$\hat{\sigma} = \sum_{i=1}^{N} \frac{w_i r_{i,F}}{N} \tag{5}$$

The initial guess of w_i and the scale estimation were obtained using the LMEDS solution. b was set to 4.8. A variant of the TUKEY M-ESTIMATOR that was found useful for the non-symmetric noise distribution was the combination of the TUKEY M-ESTIMATOR weights function w with the covariance matrix W by using $W diag(w)$ as the new weight function. See [13, 6, 17]

2 The proposed LP estimator

The uncertainty of each image point in space is a cone in $3D$. The vertex of the cone in located at the camera center, and the intersection of the cone with the image plane is the *image uncertainty circle* (or ellipse). The image uncertainty circle can be approximated by a simple polygon producing a $3D$ ray. The intersection of two rays from a stereo pair is is a convex polyhedron in $3D$ space. Fig. 1 shows the polyhedron obtained by using a rectangular uncertainty shape (The uncertainty of a pixel for example). The polyhedron shape is a function of point location in the image planes and the image uncertainty radius. This setup can also be used to express the uncertainty of point in a single image ($2D$ data) - by bounding the polygon with a global bounding cube (the room walls for example) or with a near-far planes.

Let V_i be the vertices of some polyhedron, then any point P within the polyhedron can be expressed as a convex combination:

$$\begin{aligned} P &= \sum_j V_j S_j \\ 0 &\leq S_j \leq 1, \ \sum_j S_j = 1 \end{aligned} \tag{6}$$

Given Eq. 6 The mapping of a model point M_i to the (unknown) 3D point P_i by an affine transformation F can be expressed using the bounding polyhedron V_i as:

$$\begin{aligned} FM_i &= \sum_j (V_i)_j (S_i)_j \\ 0 &\leq (S_i)_j \leq 1, \ \sum_j (S_i)_j = 1 \end{aligned} \tag{7}$$

Plugging Eq. 7 into the LAD estimator results in the following linear program, - the proposed LP pose estimator.

$$Min : \sum_i (r_i^+)_x + (r_i^-)_x + (r_i^+)_y + (r_i^-)_y + (r_i^+)_z + (r_i^-)_z \tag{8}$$

$$Subject\ to$$

$$P_i = \sum_j (V_i)_j (S_i)_j$$

$$(FM_i)_x \ + \ (r_i^+)_x - (r_i^-)_x = (P_i)_x$$

$$(FM_i)_y \ + \ (r_i^+)_y - (r_i^-)_y = (P_i)_y$$

$$(FM_i)_z \ + \ (r_i^+)_y - (r_i^-)_z = (P_i)_z$$

$$r_i^+, r_i^- \geq 0, \ \ 0 \leq (S_i)_j \leq 1, \ \ \sum_j (S_i)_j = 1$$

The value of the error function of the LP pose estimator is zero error iff F maps all model points somewhere within their corresponding polyhedron. There is no preferred point within the polyhedron - which is an advantage, especially when coping with systematic bias errors. In the case that a model point is mapped outside its corresponding polyhedron, the estimator will select the closest (in LAD sense) point within the polyhedron as the corresponding point and the L_1 distance between the selected point and the warped model point will be added to the error value.

3 Selecting the radius of image uncertainty

The radius of the image uncertainty circle is used by the WLS estimator as well as by the the proposed LP estimator. A good selection of the radius of uncertainty circle would be one that matches the uncertainty of the inliers only - due to noise. It should produce small, yet greater than zero error values for the LP estimator, since a zero error may indicate too much freedom that can cause inaccurate pose estimation. The radius of image uncertainty circle can be selected by:

1. For a given a-priory knowledge of noise parameters, the radius is according to the expectancy of the noise magnitude.
2. If no a-priory knowledge in given - but the accuracy of the estimation can be tested (like in RANSAC) then the radius can be determined by binary search over a predefined region.
3. Otherwise a robust scale estimator such as Eq. 5 can be used.

4 Tests

All tests were conducted in a simulative environment with known ground truth, known calibration and known matching for the inliers. The scale was set so that $3D$ world measurements can be regarded as millimeters and $2D$ image measurements in pixels. Simulated image resolution was 500×500 pixel, and "automatic zoom" was used to keep the model image span the full pixel range. The simulated stereo rig had base line of $150mm$, two identical parallel cameras (due to the large field of view used) and focal length of $10mm$. The field of view in the tests was 45^o and for the large field of view: 80^o. The model consisted of 100 points located on a grid volume of $4meter^3$. Model points were centered at the origin and had small additive noise added to them to break possible regularities. The pose of the model was randomly selected within rotation of $\pm 10^o$ and translation of $\pm 100mm$ Two basic types of errors were induces:

Noise - Additive, "small" magnitude (up to 4 pixels), uniform distributed, zero mean noise.

Outliers - Additive, "large" magnitude (up to full size of image in magnitude), uniform distributed, positive (non-zero mean) noise.

The proposed LP estimator (Tagged as "LP") was compared to the following algorithms:

1. Stereo reconstruction without using model. Tagged as "RAW" and given as a reference.
2. Pose estimation by the least squares estimator. Tagged as "BLS", The common frame algorithm for least squares is also presented tagged as "LSQ".
3. Pose estimation by the weighted least squares estimator, using covariance matrices. Tagged as "WLS". The covariance matrices were calculated using the same data that was used by the proposed LP estimator.
4. Pose estimation by LMEDS estimator. 500 iterations were used to give practically guaranteed outlier rejection (but not noise). Tagged as "LMedS".
5. Pose estimation by LAD estimator. Tagged as "LAD".
6. Pose estimation by TUKEY BI-WEIGHT M-ESTIMATOR. Tagged as "Tukey". The LMEDS solution was used as the initial guess for the TUKEY M-ESTIMATOR.
7. Pose estimation by a variant of the TUKEY M-ESTIMATOR that used covariance matrices as well, which produced good results in some cases. Tagged as "TWLS".

The checking criteria (compared to the ground truth before added noise and outliers) include:

1. Average and maximum absolute error between the ground truth and the warped model points according to the recovered pose. (The maximum error was selected for it implication on safety consideration in robotics).
2. Absolute difference of translation vector (for X,Y and Z).
3. Absolute difference of rotation axis (in degrees), and rotation angle about the axis (meaningful only if the difference in the rotation axis is small).

Each test was repeated several times, with different random selections each time. In cases where the results looked alike - the first result appear in the paper. In cases where different random selection cause significantly difference in the result - the first representative of each case is shown. The best results(s) in each case (judged by warped model errors), the second best result(s) and the worst result are marked. The RAW stereo data was excluded from ranking as it is given as a reference only.

4.1 Noise resistance test

In this test, additive uniform, zero mean noise was added to both images. Table 1 shows the result for maximum noise amplitude between $\frac{1}{100} \ldots 2$ pixels. We can see that:

1. Even for the lowest noise level the RAW stereo reconstruction has significant error - caused by the non optimal setting of the stereo rig.
2. The LSQ estimator did not provide the best result due to the non-Gaussian distribution of the noise.
3. The WLS and the TUKEY WLS variant estimators provided the best estimate due to use of covariance matrices.
4. The LMEDS estimator usually provided the worst results since all points had additive noise and due to its small support. (The WLS begins showing good results at about 20 points - which is already too costly for LMEDS).
5. The LP estimator provided the second best results. It was the only estimator to be at the same order of error magnitude as the best estimators. It is clearly different than the LAD estimator.

4.2 Outliers resistance test

In this test, strong, additive uniform, positive (non zero mean) noise was added to the images. The maximum noise level was 500 pixels. Different number of outliers were tested, from a single outlier to 50% outliers. Tables 2, 3 show the result for the outliers test. We can see that:

1. The LMEDS estimator and the TUKEY M-ESTIMATOR estimator (with LMEDS result used as an initial guess) provided the best result.
2. The LAD estimator provided the second best result, followed by the LP estimator.

Noise Level:		Model Error		Translation Error			Rotation Error	
1/100 Pixel	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	1.809	18.191					
	LSQ	0.113	0.302	0.001	0.031	-0.293	0.004	0.000
	BLS	0.111	0.301	0.001	0.031	-0.293	0.003	0.000
$Best >>$	WLS	0.011	0.026	0.009	0.008	0.016	0.003	0.000
	LMedS	0.633	1.874	-0.660	2.189	0.397	0.271	-0.025
	Tukey	0.066	0.243	0.079	0.220	0.115	0.025	-0.001
$Best >>$	TWLS	0.011	0.025	0.008	0.007	0.016	0.003	0.000
$2ndBest >>$	LP	0.034	0.057	0.027	0.050	0.030	0.007	0.000
	LAD	0.369	0.634	-0.029	-0.166	0.375	0.061	0.001

Noise Level:		Model Error		Translation Error			Rotation Error	
1/2 Pixel	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	92.290	963.342					
	LSQ	13.408	45.741	10.273	-55.591	-21.364	6.133	0.323
	BLS	13.564	44.147	10.273	-55.591	-21.364	5.828	0.308
$Best >>$	WLS	0.375	1.267	-0.104	0.623	0.140	0.176	0.004
	LMedS	12.234	25.086	-30.495	-8.716	-10.698	3.217	-0.011
	Tukey	10.372	35.535	-21.900	-10.043	-21.822	3.080	0.009
$Best >>$	TWLS	0.435	1.499	-0.348	0.690	0.152	0.217	0.004
$2ndBest >>$	LP	1.425	4.331	0.465	1.952	-0.850	0.093	0.022
	LAD	8.252	24.580	-13.879	-2.843	-18.381	1.793	0.020

Noise Level:		Model Error		Translation Error			Rotation Error	
1 Pixel	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	168.183	1387.741					
	LSQ	14.814	42.577	2.788	10.394	-37.582	0.814	-0.065
	BLS	14.694	42.582	2.788	10.394	-37.582	0.797	-0.067
$Best >>$	WLS	1.140	3.112	-0.651	0.840	0.798	0.222	-0.003
	LMedS	29.457	73.773	61.861	0.600	33.393	3.431	0.695
	Tukey	7.942	26.050	3.363	19.819	-15.533	1.736	-0.136
$Best >>$	TWLS	1.128	2.617	-0.804	1.583	0.862	0.113	-0.008
$2ndBest >>$	LP	2.932	9.981	0.062	-1.439	0.443	1.040	0.028
	LAD	9.444	27.931	-2.572	-19.194	7.909	3.296	0.286

Noise Level:		Model Error		Translation Error			Rotation Error	
2 Pixels	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	322.225	3487.358					
	LSQ	42.226	117.004	-2.136	22.557	-109.680	0.999	-0.143
	BLS	41.955	117.310	-2.136	22.557	-109.680	1.054	-0.148
$Best >>$	WLS	3.110	10.089	-1.587	0.521	1.056	0.887	0.012
	LMedS	66.939	158.797	139.507	2.509	-73.861	7.002	1.385
	Tukey	56.196	170.891	-44.190	38.413	-137.884	4.960	-0.376
$Best >>$	TWLS	2.930	8.785	-1.981	2.500	1.232	0.604	0.001
$2ndBest >>$	LP	7.786	18.776	5.453	17.528	-3.897	1.191	-0.032
	LAD	47.899	150.735	18.134	22.658	-120.281	5.341	-0.242

Table 1. *Noise resistance test results. See text.*

3. The least squares and weighted least squares were completely broken by the outliers.

4. The influence of the outlier on the covariance matrix overtook the TUKEY M-ESTIMATORweights causing the TUKEY M-ESTIMATOR variant to fail as well.

5. The LAD and the LP began to break down at 45%.

4.3 Combined error test

In this test, noise and outliers error were combined. Table 4 shows the result for the combined test. We can see that the LP estimator produced the best result until level of 30% outliers. (See future plans below).

4.4 Wide field of view test

The combined error test was repeated - this time for field of view of $\pm 80^o$. The results appear in Table 5. The advantage of the LP estimator increases and it produced the best results for all cases.

5 Future plans - Enhancing the LP estimator

The current LP estimator is a single iteration estimator that uses all data points with equal weights. In order to improve robustness - random sampling can be used. In order to improve accuracy - iterative re-weighting can used. Since the LP estimator have better resistance to outliers than LSQ based estimators, it is possible to allow some outliers into the data set. By doing so Equation 2 becomes:

$$1 - \left(1 - \sum_{j=1}^{r} \left(\begin{array}{c} s \\ j \end{array} \right) q^j (1-q)^{s-j} \right)^n \qquad (9)$$

where s is the number of selected points and r is the maximum number of allowed outliers. For example, it is quite feasible to select 50 points while reducing the number of outliers from 30% to 20%. Weights can be selected in a similar manner to the TUKEY M-ESTIMATOR weights. Fortunately, the standard objective function of a linear program is already formed with a weight vector: $Min: \ \ C^\top X$. We just add the vector C to the linear program. The vector C code both weights and random selection (by assigning zero weight to n on selected points). (Most linear program solvers optimize constraints with zero effect on the objective function, so the efficiency is not affected by zero weight values). Note that this section also applies to the LAD estimator.

6 Conclusions

1. As expected, the maximum likelihood estimators (s.a. WLS) produced the best estimations in the presence of noise (zero mean noise, not necessarily Gaussian). The robust estimators (such as

Outliers: 1	Estimator	Model Error Avr	Model Error Max	Translation Error X	Translation Error Y	Translation Error Z	Rotation Error Axis	Rotation Error About
	RAW	15.898	2024.944					
	LSQ	23.180	74.252	-89.455	50.472	-7.990	4.744	-1.016
	BLS	23.181	74.182	-89.455	50.472	-7.990	4.740	-1.015
	WLS	33.218	110.224	39.925	117.020	26.050	13.710	0.304
Best >>	LMedS	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Best >>	Tukey	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	TWLS	45.068	82.139	8.126	82.139	44.938	0.000	0.000
2ndBest >>	LP	0.089	0.246	-0.080	0.128	-0.070	0.031	0.001
	LAD	0.282	0.768	-0.744	-0.234	0.304	0.089	-0.001

Outliers: 10	Estimator	Model Error Avr	Model Error Max	Translation Error X	Translation Error Y	Translation Error Z	Rotation Error Axis	Rotation Error About
	RAW	226.326	6563.388					
	LSQ	160.496	454.014	186.285	-67.371	-353.152	4.125	-1.324
	BLS	161.660	450.713	186.285	-67.371	-353.152	4.025	-1.298
	WLS	3020.721	7003.426	2567.422	-265.045	-3246.323	54.197	-156.784
Best >>	LMedS	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Best >>	Tukey	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	TWLS	1523.436	3173.990	512.954	-883.363	-3173.990	0.000	0.000
	LP	0.173	0.558	-0.463	0.116	-0.274	0.027	0.007
2ndBest >>	LAD	0.002	0.004	0.003	-0.001	-0.001	0.000	0.000

Outliers: 30	Estimator	Model Error Avr	Model Error Max	Translation Error X	Translation Error Y	Translation Error Z	Rotation Error Axis	Rotation Error About
	LSQ	508.483	1471.398	-8.416	302.877	-1296.576	69.448	-0.940
	BLS	508.633	1460.329	-8.416	302.877	-1296.576	70.357	-0.794
	WLS	3286.981	9146.010	-1052.119	382.434	2836.719	23.806	-173.704
Best >>	LMedS	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Best >>	Tukey	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	TWLS	1209.244	2904.940	-434.435	288.356	-2904.940	0.000	0.000
	LP	0.486	1.165	-0.137	0.111	-1.116	0.024	0.000
2ndBest >>	LAD	0.002	0.005	-0.001	0.001	0.004	0.000	0.000

Outliers: 45	Estimator	Model Error Avr	Model Error Max	Translation Error X	Translation Error Y	Translation Error Z	Rotation Error Axis	Rotation Error About
	RAW	1014.901	6427.995					
	LSQ	705.569	2071.503	97.802	218.575	-1871.529	41.406	0.324
	BLS	713.984	2060.130	97.802	218.575	-1871.529	40.941	0.591
	WLS	3644.573	10070.096	-1219.469	-486.973	1899.952	42.737	-173.704
Best >>	LMedS	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Best >>	Tukey	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	TWLS	840.651	1907.990	-66.158	547.805	-1907.990	0.000	0.000
	LP	10.872	30.298	-0.809	0.573	-29.396	0.100	0.019
2ndBest >>	LAD	0.008	0.025	0.009	0.016	-0.006	0.003	0.000

Table 2. *Outliers resistance test results. See text.*

Outliers: 45	Estimator	Model Error		Translation Error			Rotation Error	
		Avr	Max	X	Y	Z	Axis	About
	RAW	1022.459	7436.217					
	LSQ	816.918	2743.550	486.938	1516.712	-1475.456	76.481	-16.618
	BLS	817.529	2708.373	486.938	1516.712	-1475.456	76.741	-16.275
	WLS	3859.806	11371.681	41.487	-909.021	1120.821	87.550	-171.161
$Best >>$	LMedS	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$Best >>$	Tukey	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	TWLS	692.503	1319.766	-73.745	683.999	-1319.766	90.000	0.000
	LP	189.048	734.462	231.241	674.088	-156.541	78.969	-7.978
	LAD	189.121	732.005	231.881	669.770	-161.131	79.044	-7.882

Table 3. *Outliers resistance test results. See text.*

LMEDS) produced the best estimations in the presence of outliers without any noise. The LP estimation was better than the maximum likelihood estimations in the presence of outliers and better than the robust estimations in the presence of noise and better then both in the presence of noise and outliers - especially for wide field of view.

2. The LP estimator was tested with maximum possible outlier error (in magnitude) within the frame of the image - no leverage point cases were observed - this result complies with previous results that appeared in [*].

3. The current LP estimator which is single iteration, uniform weight and includes all data points can be enhanced to exploit random sampling and iterative re-weighing within the standard framework of linear programming.

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Noise: 1/2		Model Error		Translation Error			Rotation Error	
Outliers: 10	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	330.354	6461.861					
	LSQ	211.162	720.278	244.663	-332.714	-465.684	57.474	0.137
	BLS	209.737	711.218	244.663	-332.714	-465.684	55.605	0.215
	WLS	3341.308	5777.464	-634.204	275.104	-5077.698	31.155	-172.429
	LMedS	12.949	26.111	-13.468	6.468	14.576	1.925	0.001
$2ndBest >>$	Tukey	7.904	27.983	-25.376	-14.596	11.418	3.599	0.015
	TWLS	937.255	1455.670	274.821	1080.893	-1455.206	0.087	0.009
$Best >>$	LP	2.215	7.201	-0.113	-5.457	-2.897	1.057	0.044
$2ndBest$	LAD	8.449	23.724	-12.685	-32.129	-1.661	4.989	0.148

Noise: 1		Model Error		Translation Error			Rotation Error	
Outliers: 10	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	410.494	6462.574					
	LSQ	208.935	713.489	248.853	-323.641	-460.194	56.965	0.160
	BLS	207.474	704.142	248.853	-323.641	-460.194	55.037	0.238
	WLS	3354.821	5823.751	-794.238	464.793	-5064.401	31.989	-169.974
	LMedS	75.463	204.472	-194.590	-67.718	113.134	21.553	0.601
$2ndBest >>$	Tukey	12.569	35.132	-49.394	-29.450	-2.073	7.187	0.029
	TWLS	1033.217	1709.669	307.170	1081.765	-1708.412	0.254	0.021
$Best >>$	LP	4.264	14.712	-1.869	-9.889	-6.171	1.996	0.081
	LAD	16.914	48.689	-24.899	-66.783	-4.251	10.224	0.263

Noise: 1/2		Model Error		Translation Error			Rotation Error	
Outliers: 20	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	534.974	6457.304					
	LSQ	317.323	900.667	-60.901	15.641	-868.523	6.686	0.140
	BLS	311.001	899.635	-60.901	15.641	-868.523	6.717	0.041
	WLS	3478.124	5790.135	-191.580	-322.390	-5459.968	34.674	-171.707
	LMedS	27.519	83.162	-64.810	107.649	9.294	8.835	-1.079
$2ndBest >>$	Tukey	8.830	28.039	11.958	0.238	-21.447	1.666	0.014
	TWLS	1037.784	2158.402	262.833	691.552	-2157.532	0.200	-0.014
$Best >>$	LP	3.745	10.453	-4.840	0.323	-7.248	0.751	-0.015
	LAD	24.196	77.657	17.441	-13.986	-59.051	3.699	0.178

Noise: 1/2		Model Error		Translation Error			Rotation Error	
Outliers: 30	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	760.803	6462.767					
	LSQ	524.166	1726.181	-578.041	-433.829	-1216.117	70.138	-3.891
	BLS	520.750	1712.118	-578.041	-433.829	-1216.117	68.799	-3.778
	WLS	4472.985	8330.073	-1299.379	-1295.965	-8079.397	33.428	-167.652
$2ndBest >>$	LMedS	16.647	59.631	54.875	3.462	-29.917	8.572	0.000
$Best >>$	Tukey	14.425	49.392	-1.603	-65.567	-18.837	8.063	0.376
	TWLS	789.280	1256.991	444.825	1256.446	-664.966	0.248	0.007
	LP	26.023	92.428	-70.492	-39.799	-44.107	10.513	0.202
	LAD	39.923	135.584	-65.520	-31.254	-84.921	11.804	0.179

Table 4. *Combined Noise and Outliers resistance test results. See text.*

Noise: 1/2		Model Error		Translation Error			Rotation Error	
Outliers: 10	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	540.999	6085.128					
	LSQ	149.257	494.011	144.449	-122.759	-328.790	36.367	0.671
	BLS	149.452	485.989	144.449	-122.759	-328.790	34.432	0.672
	WLS	2311.542	6197.539	-1076.314	1203.634	-1669.289	30.416	-160.070
	LMedS	136.350	308.560	66.910	-31.657	-280.050	5.013	-0.835
	Tukey	43.952	124.268	-58.412	40.524	-78.515	7.686	-0.400
	TWLS	1427.543	2133.874	-728.787	2122.566	-1424.366	1.362	0.078
$Best >>$	LP	8.570	23.198	-4.450	4.890	-8.686	2.441	0.118
$2ndBest >>$	LAD	24.561	65.739	-2.359	-3.416	-37.169	4.548	-0.544

Noise: 1		Model Error		Translation Error			Rotation Error	
Outliers: 10	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	1010.969	26429.873					
	LSQ	554.204	2049.524	812.911	-673.969	-877.471	75.363	-12.393
	BLS	548.012	1982.529	812.911	-673.969	-877.471	76.089	-11.444
	WLS	2435.348	6633.286	-1186.128	2156.349	-1824.252	29.462	-158.329
	LMedS	278.094	863.033	-127.341	-50.840	-665.746	43.113	0.046
	Tukey	477.633	1747.181	628.401	-523.574	-808.699	75.444	-8.878
	TWLS	1587.257	2959.155	-216.615	2922.414	-1615.280	5.196	0.056
$Best >>$	LP	42.467	125.347	3.214	17.726	-104.093	4.168	-0.278
	LAD	149.153	446.592	-13.903	9.976	-329.151	78.027	-2.051

Noise: 1/2		Model Error		Translation Error			Rotation Error	
Outliers: 20	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	644.872	7978.731					
	LSQ	222.049	704.285	38.317	-86.282	-595.499	25.763	0.750
	BLS	218.571	688.169	38.317	-86.282	-595.499	22.022	0.708
	WLS	3290.707	7875.635	-3935.027	-2673.111	-2663.154	28.169	-168.929
	LMedS	187.139	469.149	58.104	24.594	-410.256	12.638	-0.308
	Tukey	1295.181	2464.983	949.616	-1625.748	1940.417	45.014	-4.116
	TWLS	2602.932	5943.315	958.470	5937.370	-909.445	0.749	0.026
$Best >>$	LP	31.054	111.478	-45.373	27.961	-50.135	12.499	-0.273
$2ndBest >>$	LAD	82.355	243.906	-14.799	21.514	-219.603	6.353	0.147

Noise: 1/2		Model Error		Translation Error			Rotation Error	
Outliers: 30	Estimator	Avr	Max	X	Y	Z	Axis	About
	RAW	749.124	4368.621					
	LSQ	311.688	998.178	-350.897	-116.254	-695.196	50.792	-2.239
	BLS	310.007	992.931	-350.897	-116.254	-695.196	50.034	-2.296
	WLS	2751.972	7107.815	-3003.917	-1.453	-2270.162	35.341	-171.986
	LMedS	218.722	778.933	-325.832	440.850	-114.510	60.658	-8.958
	Tukey	110.744	342.292	-97.494	138.649	228.195	16.395	-1.663
	TWLS	1964.493	4265.691	448.350	4255.876	-1184.232	0.971	0.077
$Best >>$	LP	62.339	216.881	-114.129	26.686	-94.911	25.224	-1.053
$2ndBest >>$	LAD	86.095	280.483	-45.449	8.512	-217.496	13.088	-0.300

Table 5. *Wide FOV test test results. See text.*

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